Thermal Phases beyond Spontaneous Symmetry Breaking

Classification of phases beyond symmetry at finite temperature

Haruki Watanabe

Department of Physics & Institute for Advanced Study, HKUST

with Sena Watanabe, Yukitoshi Motome (U. Tokyo); Shunsuke Sengoku, Weiguang Cao, Guanxi Xun (HKUST)

This talk is based on

I. S. Watanabe, Y. Motome, H. Watanabe, PRB 114, 094108 (2026); PRL in print (arXiv:2603.03852); PRX 16, 041016 (2026); JPSJ 95, 085002 (2026)

II. H. Watanabe, arXiv:2607.00134

III. S. Sengoku, W. Cao, H. Watanabe, arXiv:2610.27938; G. Xun, W. Cao, H. Watanabe, arXiv:2610.27939

What distinguishes two phases of matter?

At $T = 0$: classification goes beyond symmetry

- intrinsic topological order
- symmetry-protected topological phases
- quantized responses

At $T > 0$: the Landau paradigm

- phases $\leftrightarrow$ patterns of spontaneous symmetry breaking
- local order parameters
- same symmetry $\Rightarrow$ connectable?

Do Gibbs states $\rho = e^{-\beta H}/Z$ at $T > 0$ admit a classification **beyond symmetry breaking**?

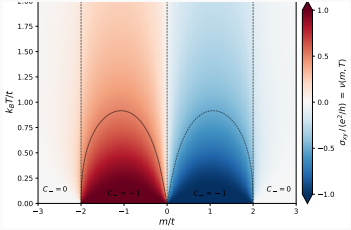
Landau, Zh. Eksp. Teor. Fiz. 7, 19 (1937); Wen, Int. J. Mod. Phys. B 4, 239 (1990); Kitaev, Ann. Phys. 303, 2 (2003); Chen, Gu, Liu, Wen, PRB 87, 155114 (2013)

Example: a Chern insulator at $T > 0$ loses its topological distinction

Qi–Wu–Zhang model (two-band Chern insulator): $h_{\boldsymbol{k}} = \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}$, $\boldsymbol{g}_{\boldsymbol{k}} = (\sin k_x,\ \sin k_y,\ m - \cos k_x - \cos k_y)$

$\sigma_{xy}(T) = \frac{e^2}{\hbar A} \sum_{\boldsymbol{k}} \sum_n f(\epsilon_{n\boldsymbol{k}})\, \Omega_n(\boldsymbol{k})$, $\quad \Omega_n(\boldsymbol{k}) = -2\,\mathrm{Im}\langle \partial_{k_x} u_{n\boldsymbol{k}} | \partial_{k_y} u_{n\boldsymbol{k}} \rangle$ (Kubo; f: Fermi function)

- $T \to 0$: $\sigma_{xy} = \frac{e^2}{h} C_-$ with $C_- = -\mathrm{sgn}(m)$ for $0 < |m| < 2$ and $C_- = 0$ for $|m| > 2$: distinct phases separated by gap closings
- $T > 0$: σ_{xy} is no longer quantized and the free energy is analytic in m: Chern and trivial insulators are **connected**
- SPT order protected by an on-site symmetry also cannot persist at $T > 0$; with a 1-form symmetry it can

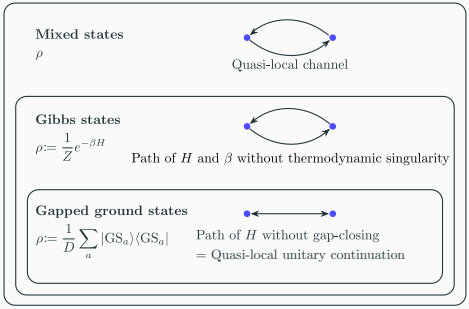


$\nu(m, T) = \sigma_{xy}(T)/(e^2/h)$ at $\mu = 0$; dashed: $T = 0$ gap closings

Ground-state topological invariants do not label thermal states.
Goal: a topological invariant that distinguishes phases of Gibbs states at $T > 0$.

What does “the same phase” mean for a mixed state?

- gapped ground states: local unitary / quasi-adiabatic continuation
- general mixed states: **two-way local quantum channels** — $\rho \sim \sigma$ iff local channels with product ancillas and causal radius $o(L)$ convert $\rho \to \sigma$ **and** $\sigma \to \rho$ with trace-norm error $\to 0$
- Gibbs states: paths without thermodynamic singularity



The classification of mixed-state phases is still **developing**, with several competing definitions. Whether the two-way local-channel definition and free-energy analyticity single out the same phases is **not known**.

Hastings & Wen, PRB 72, 045141 (2005); Chen, Gu, Wen, PRB 82, 155138 (2010); Coser & Pérez-García, Quantum 3, 174 (2019); Sang, Zou, Hsieh, PRX 14, 031044 (2024)

Plan of the talk

I. **Phases of water.** Ice VII, VIII, X and spin ice: a case where symmetry is enough
monopoles open the loops $\Rightarrow$ crossover

II. **3D toric code in magnetic fields.** A thermal phase beyond symmetry
closed flux loops $\Rightarrow$ an invariant $\mathcal{F}$ of two-way local-channel phases

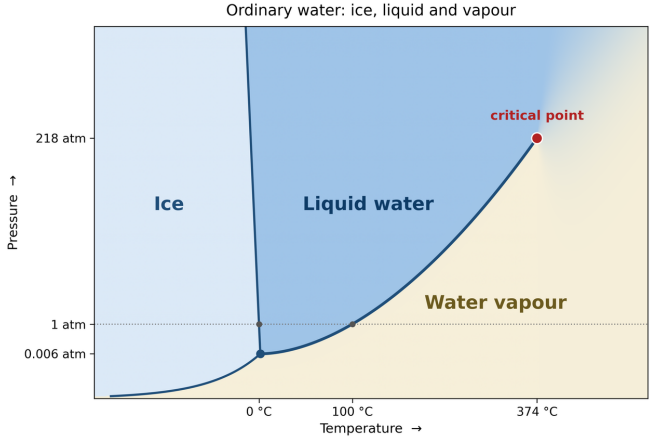
III. **Classical $\mathbb{Z}_2$ models on m-cells.** Stability theorems in general (d, m)
free-energy analyticity and local-channel equivalence without symmetry protection

Part I: Phases of water — where symmetry is enough

The first phase diagram we ever learn

- Ice, liquid water, vapour
- The liquid–vapour line **ends** at a critical point ($374\,^{\circ}$C, 218 atm)
- Beyond it, liquid and vapour transform into each other **continuously**

Liquid water and steam are **one** thermodynamic phase.



Stanley, *Introduction to Phase Transitions and Critical Phenomena* (1971); Chaikin & Lubensky, *Principles of Condensed Matter Physics* (1995)

Why can the liquid–vapour line end, but not the melting line?

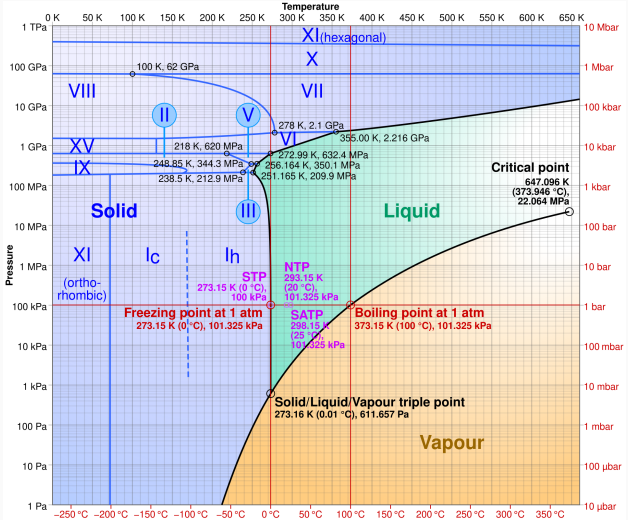
	broken symmetry	local order parameter
vapour	none	—
liquid	none	—
ice	translations, rotations	crystalline density wave

- Two states with the **same** symmetry may be connected without a singularity
- Two states with **different** symmetry are always separated by a phase boundary

Landau–Ginzburg: at $T > 0$, phases are classified by **spontaneous symmetry breaking**. Is this complete?

Landau (1937); Anderson, *Basic Notions of Condensed Matter Physics* (1984)

The phase diagram of water



Cmglee, "Phase diagram of water," Wikimedia Commons, CC BY-SA 3.0 (log–lin chart, 1 Pa–1 TPa, 0–660 K; data compiled from Chaplin, *Water Structure and Science*, lsbu.ac.uk/water)

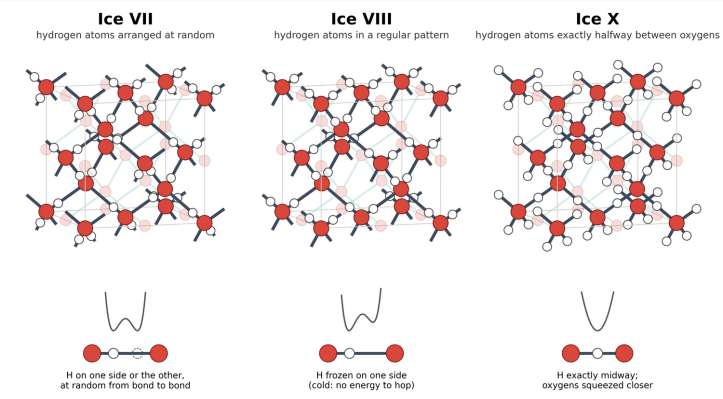
The icefields: more than twenty crystalline phases

- Ice I_h, I_c, II, III, ..., XX: from negative pressure to > 100 GPa
- Almost all are distinguished by their **space group** $\Rightarrow$ the Landau classification applies
- Exceptions that share one space group: **ice VII and ice X** ($Pn\bar{3}m$), ice III and IX ($P4_12_12$)

Are ice VII and ice X different phases, or the same phase in different regimes?

Bartels-Rausch *et al.*, Rev. Mod. Phys. 84, 885 (2012); Salzmann *et al.*, Phys. Chem. Chem. Phys. 13, 18468 (2011); Hansen, Nat. Commun. 12, 3161 (2021)

Dense ice: one oxygen framework, three hydrogen arrangements



- **VII**: protons disordered on two off-center sites per bond (ice rules) — $Pn\bar{3}m$
- **VIII**: protons ordered, antiferroelectric — $I4_1/amd$, cubic symmetry broken
- **X**: protons at the bond midpoints — $Pn\bar{3}m$, **same space group as VII**

Kuhs *et al.*, J. Chem. Phys. 81, 3612 (1984); Hemley *et al.*, Nature 330, 737 (1987); Holzapfel, J. Chem. Phys. 56, 712 (1972)

What experiments say

Low temperature: VIII $\rightarrow$ X is first order (hysteresis)

Room temperature: VII $\rightarrow$ X is a broad, continuous regime

- infrared and Raman spectroscopy
- high-pressure NMR
- in situ neutron diffraction: H-bond symmetrisation

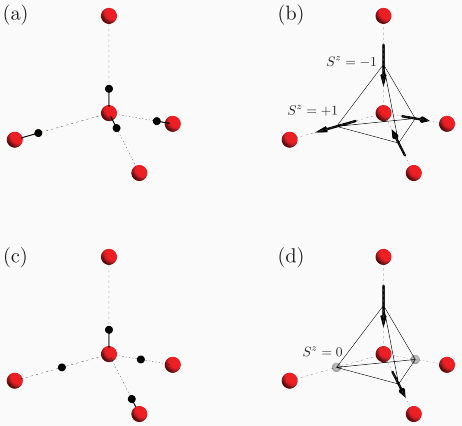
Thermodynamic model: a VII–X first-order line ending at a critical point hidden inside the VIII region

No symmetry distinguishes VII from X.

A sharp boundary at $T > 0$ would have to be **topological**.

Hirsch & Holzapfel (1986); Goncharov *et al.*, Science 273, 218 (1996); Meier *et al.* (2018); Komatsu *et al.*, Nat. Commun. 15, 5100 (2024); Ackland *et al.*, PRB 112, 094116 (2025)

From protons to spins: a spin-1 pyrochlore magnet



- O–O bond midpoints form a **pyrochlore lattice**
- Proton near A / at midpoint / near B $\iff S^z_\ell = -1, 0, +1$
- Ice rules $\iff$ "two-in, two-out"
- Ice-rule violation $(\mathrm{H_3O^+},\ \mathrm{OH^-}) \iff$ **magnetic monopole**

Dense ice = classical **spin-1 spin ice**

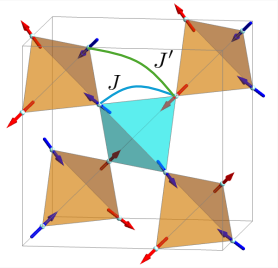
Harris *et al.*, PRL 79, 2554 (1997); Bramwell & Gingras, Science 294, 1495 (2001); Castelnovo, Moessner, Sondhi, Nature 451, 42 (2008); Pandey & Damle, arXiv:2512.11623

The effective model

$$H = J\sum_{\langle \ell\ell' \rangle} S^z_\ell S^z_{\ell'} + J'\sum_{\{\ell\ell'\}} S^z_\ell S^z_{\ell'} + \Delta \sum_\ell (S^z_\ell)^2$$

For $J' = 0$ and $\Delta = J + \mu$:

$$H = \frac{J}{2}\sum_{\boldsymbol{r}} \Big(\sum_{\ell \in \boldsymbol{r}} S^z_\ell \Big)^2 + \mu \sum_\ell (S^z_\ell)^2$$



- Δ: cost of off-center protons $\rightarrow$ drives ice X
- $J' > 0$: selects the polarized state $\rightarrow$ ice VIII
- monopole charge $Q_{\boldsymbol{r}} = \nabla \cdot \boldsymbol{S}_{\boldsymbol{r}} := \pm \sum_{\ell \in \boldsymbol{r}} S^z_\ell$ ($+$ on A, $-$ on B sites of the diamond lattice) costs $\frac{J}{2}Q^2$

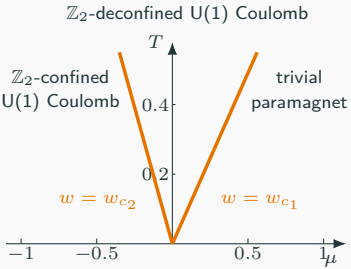
$w := e^{-\mu/T}$ (spin ± 1 vs. 0), $v := e^{-J/2T}$ (**monopoles**):

$$Z = \sum_{\{S^z_\ell\}} e^{-\beta H} = \sum_{\{Q_{\boldsymbol{r}}\}} \sum_{\{S^z_\ell\}} \prod_{\boldsymbol{r}} \delta_{\nabla \cdot \boldsymbol{S}_{\boldsymbol{r}}, Q_{\boldsymbol{r}}}\, v^{\sum_{\boldsymbol{r}} Q^2_{\boldsymbol{r}}}\, w^{\sum_\ell (S^z_\ell)^2}$$

Monopole-free limit $v = 0$: three phases, no symmetry breaking

$\boldsymbol{P} \coloneqq \frac{1}{L}\sum_{\ell=\langle \boldsymbol{r}_A \boldsymbol{r}_B\rangle}(\boldsymbol{r}_B - \boldsymbol{r}_A)\, S^z_\ell$ (net S^z flux through a plane; ice rule at $v = 0 \Rightarrow \boldsymbol{P} \in \mathbb{Z}^3$)

	$w < w_{c_1}$ trivial PM	$w_{c_1} < w < w_{c_2}$ $\mathbb{Z}_2$-deconfined	$w > w_{c_2}$ $\mathbb{Z}_2$-confined
ice	**ice X**	—	**ice VII**
emergent photon	gapped	gapless	gapless
flux sectors $\boldsymbol{P}$	$\mathbf{0}$	$\mathbb{Z}^3$	$(2\mathbb{Z})^3$
smallest free charge	—	$Q = 1$	$Q = 2$
transition	w_{c_1}: 3D XY		w_{c_2}: 3D Ising



$v = 0$ boundaries (Pandey & Damle):
$w = e^{-\mu/T} = w_c$, i.e. $\mu = -T \ln w_c$

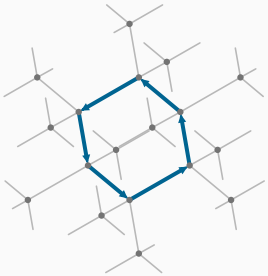
At $v = 0$, ice VII and ice X **are** distinct phases, separated by two topological transitions.

Duality I (small w): directed spin loops $\Rightarrow$ 3D XY model

$S^z = \pm 1$ links are unit currents in an $S^z = 0$ background. Gauss law at $v = 0 \Rightarrow$ **directed closed loops**. Write each constraint as $\delta_{\nabla\cdot \boldsymbol{S}_{\boldsymbol{r}},0} = \int_0^{2\pi} \frac{d\theta_{\boldsymbol{r}}}{2\pi}\, e^{-i\theta_{\boldsymbol{r}}\nabla\cdot \boldsymbol{S}_{\boldsymbol{r}}}$:

$$
\begin{aligned}
Z_{v=0} &= \textstyle\sum_{\{S^z_\ell\}} \prod_{\boldsymbol{r}} \delta_{\nabla\cdot \boldsymbol{S}_{\boldsymbol{r}},\,0}\; w^{\sum_\ell (S^z_\ell)^2} \\
&= \textstyle\int \mathcal{D}\theta \sum_{\{S^z_\ell\}} e^{-i\sum_{\boldsymbol{r}} \theta_{\boldsymbol{r}}\nabla\cdot \boldsymbol{S}_{\boldsymbol{r}}}\; w^{\sum_\ell (S^z_\ell)^2} \\
&= \textstyle\int \mathcal{D}\theta \prod_\ell \left[\sum_{S^z_\ell = 0,\pm 1} w^{(S^z_\ell)^2} e^{iS^z_\ell \nabla\theta_\ell}\right] \\
&= \textstyle\int \mathcal{D}\theta \prod_\ell \left[1 + 2w\cos\nabla\theta_\ell\right] \approx \int \mathcal{D}\theta\, e^{2w\sum_\ell \cos\nabla\theta_\ell},
\end{aligned}
$$

$\nabla\theta_\ell = \theta_{\boldsymbol{r}_B} - \theta_{\boldsymbol{r}_A}$ ($\sum_{\boldsymbol{r}} \theta_{\boldsymbol{r}}\nabla\cdot\boldsymbol{S}_{\boldsymbol{r}} = -\sum_\ell S^z_\ell \nabla\theta_\ell$): 3D XY, $K = 2w$.
T_c^{XY}(diamond) $= 1.30036(1) \Rightarrow w_{c_1} \simeq 0.385$ (MC: 0.3609)

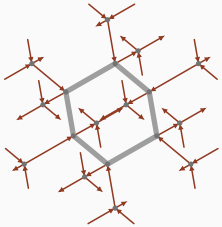


Loop proliferation at w_{c_1} = ordering of the dual XY model, $\langle e^{i\theta}\rangle \neq 0$

José, Kadanoff, Kirkpatrick, Nelson, PRB 16, 1217 (1977); Savit, Rev. Mod. Phys. 52, 453 (1980); S. Watanabe, Motome, H. Watanabe, J. Phys. Soc. Jpn. 95, 085002 (2026); PRL in print (arXiv:2603.03852)

Duality II (large w): vacancy loops $\Rightarrow$ 3D Ising model

$S^z = 0$ links are unoriented vacancies in an $S^z = \pm 1$ ice background.
At $v = 0$ every site has an even number of them $\Rightarrow$ **undirected closed loops** C, each with $W(C)$ ice-rule backgrounds.



$$Z_{v=0} = \sum_{\{S^z_\ell\}} \prod_{\boldsymbol{r}} \delta_{\nabla\cdot \boldsymbol{S}_{\boldsymbol{r}},\,0}\; w^{\sum_\ell (S^z_\ell)^2} = \sum_C w^{2N-|C|}\, W(C)$$
$$= w^{2N} W(\emptyset) \sum_C \frac{W(C)}{W(\emptyset)}\, w^{-|C|} \overset{\text{Pauling}}{\approx} w^{2N} W(\emptyset) \sum_C \left(\tfrac{2}{3w}\right)^{|C|}$$

Pauling: $W(\emptyset) \approx 2^{2N}(3/8)^N$ and $W(C) \approx 2^{2N-|C|}(3/8)^{N-|C|}(1/2)^{|C|} \Rightarrow W(C)/W(\emptyset) \approx (2/3)^{|C|}$.
Compare with the **high-temperature expansion of the 3D Ising model** on the diamond lattice:

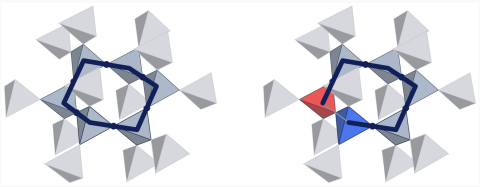
$$Z_{\text{Ising}} = \sum_{\{\sigma_{\boldsymbol{r}}\}} \prod_{\langle \boldsymbol{r}\boldsymbol{r}'\rangle} e^{K\sigma_{\boldsymbol{r}}\sigma_{\boldsymbol{r}'}} = \sum_{\{\sigma_{\boldsymbol{r}}\}} \prod_{\langle \boldsymbol{r}\boldsymbol{r}'\rangle} \left(\cosh K + \sigma_{\boldsymbol{r}}\sigma_{\boldsymbol{r}'} \sinh K\right) = (\cosh K)^{2N}\, 2^N \sum_C (\tanh K)^{|C|}$$

Ising critical coupling on the diamond lattice $K_c \simeq 0.3698 \Rightarrow w_{c_2} \simeq 1.884$ (MC: 1.8904)

Loop proliferation at w_{c_2} = ordering of the dual Ising model, $\langle \sigma_{\boldsymbol{r}} \rangle \neq 0$

Wegner (1971); Nahum *et al.*, PRL 107, 110601 (2011); Deng & Blöte, PRE 68, 036125 (2003); loop parities in the 3D toric-code limit of the Kitaev model: Nasu *et al.*, PRB 89, 115125 (2014)

Finite temperature: thermal monopoles open the loops



XY **picture** (small w)

$$Z \approx \int \mathcal{D}\theta \, e^{2w \sum_{\ell} \cos \nabla\theta_{\ell} \, + \, 2v \sum_{\boldsymbol{r}} \cos \theta_{\boldsymbol{r}}}$$

a uniform field $h = 2v$ on the dual XY spins

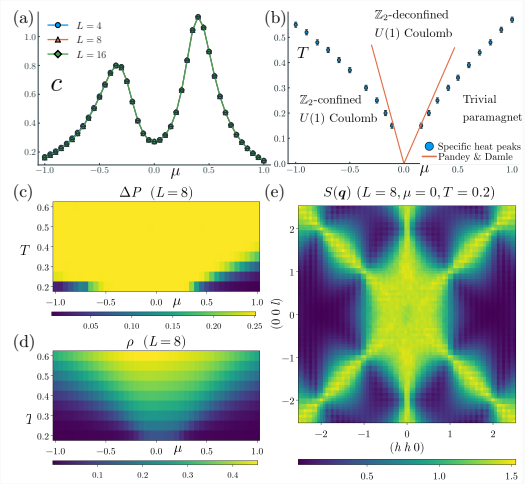
Ising picture (large w)

$$Z \approx \sum_{\{\sigma_{\boldsymbol{r}}\}} e^{K \sum_{\langle \boldsymbol{r}\boldsymbol{r}' \rangle} \sigma_{\boldsymbol{r}}\sigma_{\boldsymbol{r}'} + h \sum_{\boldsymbol{r}} \sigma_{\boldsymbol{r}}}$$

the Ising model in a field, $\tanh K = 2/(3w)$, $\tanh h = \sqrt{3}\, v$: the monopole fugacity is the field

Thermal monopoles = a **symmetry-breaking field** in the dual theory: every continuous transition becomes a **crossover** at any $T > 0$ (the field is relevant, $y_h > 0$).

Monte Carlo: rounded peaks, no quantized flux



(a) specific heat: $L = 4, 8, 16$ coincide, no divergence

(b) peak positions drift from the $v = 0$ boundaries: a **crossover diagram**

(c) global flux $\boldsymbol{P}$ leaks away from integers

(d) monopole density is finite

(e) pinch points persist at low T (long screening length)

At $T > 0$ the topological distinctions of the $v = 0$ model are gone.

Back to ice: the full model with J' and thermal monopoles

Classical Monte Carlo of the spin-1 Blume–Capel model on the pyrochlore lattice ($N = 16L^3$ sites, $J = 1$)

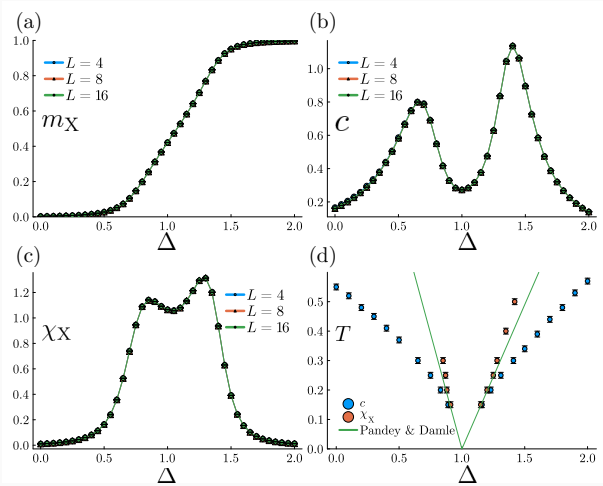
- heat-bath + short-loop updates, replica exchange in Δ
- ice-X indicator: $m_{\mathrm{X}} = 1 - \frac{1}{N}\sum_{\ell}\langle (S^z_\ell)^2\rangle$
- ice-VIII order parameter: $m_{\mathrm{VIII}} = \langle \boldsymbol{M}^2\rangle$ (polarization)
- specific heat c and susceptibility χ_{X}

Test: does any second derivative of the free energy diverge between VII and X?

Ice VII $\rightarrow$ ice X is a crossover

- m_{X} rises smoothly with Δ ($T = 0.3$)
- peaks of c and χ_{X} **saturate**: $L = 4, 8, 16$ coincide
- peak positions of c and χ_{X} **do not coincide** and drift from the $v = 0$ boundaries
- the only singularity is the $T = 0$ level crossing at $\Delta = J$

No thermodynamic singularity between VII and X at any $T > 0$.

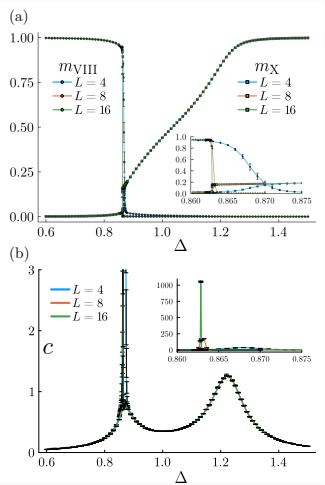


Ice VIII is bounded by a genuine first-order transition

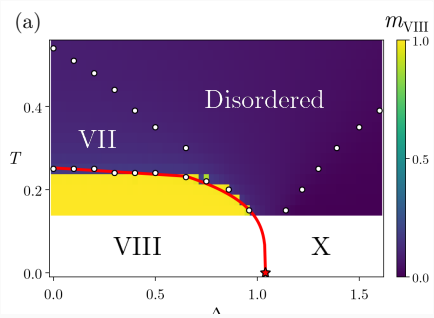
$J' = 0.02,\ T = 0.2$:

- m_{VIII} drops discontinuously at $\Delta \simeq 0.86$
- the specific-heat peak grows $\propto L^3$: first order
- m_{X} jumps but does not saturate: VIII $\to$ disordered $\to$ X
- VIII breaks the cubic symmetry (polarization along one axis): a **symmetry-protected boundary**

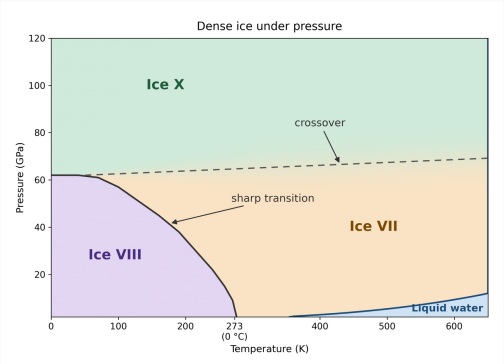
Symmetry breaking $\Rightarrow$ sharp boundary.
No symmetry breaking $\Rightarrow$ crossover.



Phase diagram of dense ice



model (m_{VIII}), $J' = 0.02$

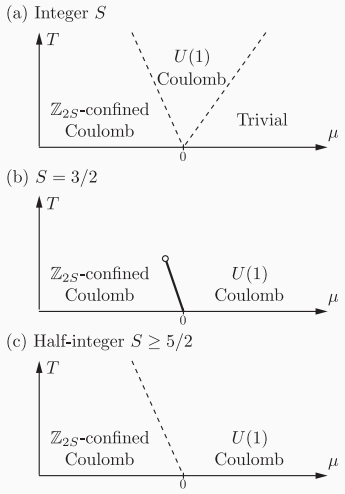


schematic

Ice VIII: a first-order line. Ice VII and ice X: one phase joined by a crossover through a disordered regime, consistent with room-temperature experiments.

S. Watanabe, Motome, H. Watanabe, PRB 114, 094108 (2026); Goncharov *et al.* (1996); Komatsu *et al.* (2024); Ackland *et al.*, PRB 112, 094116 (2025)

Arbitrary spin S: which transitions survive the monopoles?



Monopole-free limit ($T/J \to 0$ at fixed μ/T: rays from the origin)

- integer S: trivial $\to$ U(1) Coulomb (3D XY) $\to \mathbb{Z}_{2S}$-confined Coulomb, flux quantized in units of $2S$
- half-integer S: no trivial phase; $S = \frac{3}{2}$ maps to the 3-state Potts model $\Rightarrow$ **first order**; $S \geq \frac{5}{2}$: 3D XY

Finite temperature: thermal monopoles are string endpoints, a field $h_{\text{eff}} \simeq \sqrt{3}\, v$ for every S

- continuous transitions become crossover ridges (dashed lines)
- the $S = \frac{3}{2}$ first-order line survives as a coexistence line (solid) ending at a **critical end point** (open circle), at zero external field

Like liquid–gas: a first-order line survives a weak field; a continuous transition cannot.

S. Watanabe, Motome, H. Watanabe, "Topological phase transitions and their thermodynamic fate in arbitrary-S classical pyrochlore spin ice," Phys. Rev. X 16, 041016 (2026); monopole-free $S = 3/2$: Pandey, Kundu, Damle, arXiv:2602.23041

Part I: summary

- Water and steam: one phase. Ice VII and ice X: one phase at $T > 0$.
- The $v = 0$ topological transitions are real, but **point-like monopoles open the loops** and act as a symmetry-breaking field in the dual theory $\Rightarrow$ crossover.
- Ice VIII is protected by broken symmetry; its boundary is sharp. For general S, only the first-order $S = \frac{3}{2}$ line can survive, ending at a critical end point.

For water, classification by spontaneous symmetry breaking is **complete**: wherever symmetry could not distinguish two states, a continuous path connected them.

To go beyond symmetry we need defects that **cannot** break. $\Rightarrow$ Part II

Part II: 3D toric code in fields — a thermal phase beyond symmetry

Open loops vs. closed loops: the principle

Spin ice / dense ice

the ice rule is **energetic**, cost $\frac{J}{2}Q^2$

thermal monopoles cut the loops

$\Rightarrow$ relevant dual ordering field

$\Rightarrow$ the transition becomes a **crossover**

3D toric code / $\mathbb{Z}_2$ gauge theory

Bianchi identity $\prod_{p\subset\partial c} B_p = \mathbf{1}$

flux loops are closed **at every** T

$\Rightarrow$ line tension vs. entropy

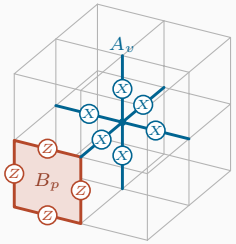
$\Rightarrow$ the thermal transition **survives**

Whether defect loops can break decides whether a thermal transition without symmetry breaking can exist.
In ice they break. In the 3D toric code they cannot — even when every microscopic symmetry is broken.

Castelnovo & Chamon, PRB 78, 155120 (2008); Nasu *et al.*, PRB 89, 115125 (2014); Fradkin & Shenker, PRD 19, 3682 (1979); S. Watanabe, Motome, H. Watanabe, PRL in print (arXiv:2603.03852)

The model: 3D toric code with both fields

$$H = -J_e \sum_v A_v - J_m \sum_p B_p - h_x \sum_\ell X_\ell - h_z \sum_\ell Z_\ell, \qquad A_v = \prod_{\ell \ni v} X_\ell, \quad B_p = \prod_{\ell \in \partial p} Z_\ell$$



qubits on the links of a cubic lattice

$J_e = J_m = 1$; the two fields:

- h_z breaks the electric 1-form symmetry $U_e(\Sigma^*) = \prod_{\ell \perp \Sigma^*} X_\ell$
- h_x breaks the magnetic 2-form symmetry $U_m(C) = \prod_{\ell \in C} Z_\ell$
- both fields: **no microscopic higher-form symmetry remains**

Known limits:

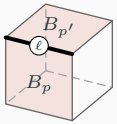
- $h = 0$: thermal transition $T_c^{(0)} \simeq 1.3133$ (3D Ising); TEE $= 2 \ln 2$ $(T = 0)$, $\ln 2$ $(0 < T < T_c^{(0)})$, 0 $(T > T_c^{(0)})$
- $T = 0$: first order at $h_x = 1$; continuous at $h_z \simeq 0.194$

Both microscopic symmetries explicitly broken. What, if anything, survives?

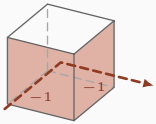
Wegner (1971); Castelnovo & Chamon, PRB 78, 155120 (2008); Reiss & Schmidt, SciPost Phys. 6, 078 (2019); Gaiotto, Kapustin, Seiberg, Willett, JHEP 02, 172 (2015); Bravyi, Hastings, Michalakis, J. Math. Phys. 51, 093512 (2010)

What survives: the closed-flux constraint

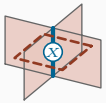
Bianchi identity on every cube c: $\prod_{p \subset \partial c} B_p = \mathbf{1}$



(a) each link ℓ of c lies in exactly two faces p, p', and $Z_\ell^2 = \mathbf{1}$



(b) so each cube has an *even* number of $B_p = -1$ faces (shaded): flux lines cannot end



(c) X_ℓ flips the four plaquettes around ℓ: the smallest flux loop

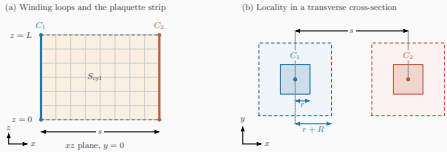
- plaquette defects ($B_p = -1$) have no endpoints: the dual edges piercing them form **closed flux loops**
- an **operator identity**: it holds at every T, h_x, h_z and for any perturbation — unlike the energetic ice rule, there is no endpoint mechanism
- low T: loops have tension; high T: loops proliferate

A constraint, not a symmetry, survives the explicit symmetry breaking — and it is exactly what a **decoder** needs.

The decoded Wilson-loop correlation f_W

$$f_W := \langle W_1 W_2 \rangle_{\mathcal{C}(\rho)} - \langle W_1 \rangle_{\mathcal{C}(\rho)} \langle W_2 \rangle_{\mathcal{C}(\rho)} = \langle \widetilde{W}_1 \widetilde{W}_2 \rangle_\rho - \langle \widetilde{W}_1 \rangle_\rho \langle \widetilde{W}_2 \rangle_\rho \qquad \widetilde{W}_i := \mathcal{C}^\dagger(W_i)$$

- two parallel winding loops C_1, C_2 at separation $s = \lfloor L/2 \rfloor$; $W_i = \prod_{\ell \in C_i} Z_\ell$, $W_1 W_2 = \prod_{p \in S_{\mathrm{cyl}}} B_p$
- a small flux loop near the strip flips the bare product $\Rightarrow$ first apply a **local recovery** $\mathcal{C}$: measure the syndrome η, flip the link set $\mathcal{R}(\eta)$ chosen by a fixed local decoder of depth $r = \lfloor \sqrt{L} \rfloor \ll s$



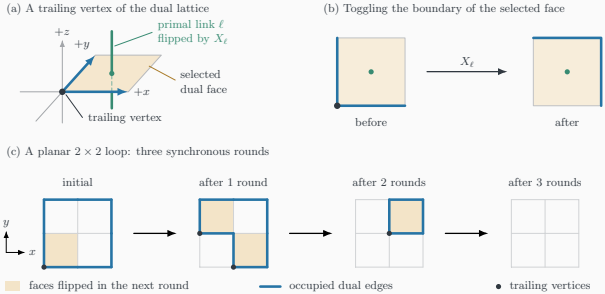
$$\mathcal{C}(\rho) := \sum_\eta X_{\mathcal{R}(\eta)} P_\eta \, \rho \, P_\eta X_{\mathcal{R}(\eta)}, \qquad P_\eta := \prod_p \tfrac{\mathbf{1} + \eta_p B_p}{2}, \qquad X_{\mathcal{R}(\eta)} := \prod_{\ell \in \mathcal{R}(\eta)} X_\ell$$

$$\langle W_i \rangle_{\mathcal{C}(\rho)} = \mathrm{Tr}\left[W_i \sum_\eta X_{\mathcal{R}(\eta)} P_\eta \, \rho \, P_\eta X_{\mathcal{R}(\eta)} \right] = \mathrm{Tr}\left[\widetilde{W}_i \, \rho \right] \quad \text{(cyclicity of the trace, } [P_\eta, W_i] = 0)$$

$$\widetilde{W}_i = \mathcal{C}^\dagger(W_i) = \sum_\eta P_\eta X_{\mathcal{R}(\eta)} W_i X_{\mathcal{R}(\eta)} P_\eta = W_i \sum_\eta (-1)^{|\mathcal{R}(\eta) \cap C_i|} P_\eta \quad (X_\ell W_i X_\ell = -W_i \text{ for } \ell \in C_i)$$

$\widetilde{W}_i$ is diagonal in the Z basis $\Rightarrow$ computable from QMC snapshots even at $h_x \neq 0$.

The fixed local decoder: Sweep



- a *trailing* dual vertex has occupied edges only in positive directions $\Rightarrow$ flip the link piercing the square they span
- synchronous rounds; r rounds erase any isolated component with $\leq 2(r+1)$ edges
- $w_i(z)$ depends only on spins within distance r of C_i: the two decoded observables live in **disjoint tubes**

Why the connected subtraction matters

$$D_{12} \coloneqq \langle \widetilde{W}_1 \widetilde{W}_2 \rangle_\rho, \quad M_i \coloneqq \langle \widetilde{W}_i \rangle_\rho, \quad f_W = D_{12} - M_1 M_2, \quad |f_W| \le \sqrt{(1-M_1^2)(1-M_2^2)} \le 1$$

shared random winding bit

$D_{12} = 1,\ M_i = 0 \Rightarrow f_W = 1$

e.g. the canonical $T = 0$ state $P_{0,L}/8$ (all 8 sectors)

pinned by polarization

$D_{12} = 1,\ M_1 M_2 = 1 \Rightarrow f_W = 0$

e.g. all $Z_\ell = +1$ (no syndrome, no correction)

f_W detects a **locally recoverable, unbiased, shared** bit — exactly what a product state cannot carry.

Theorem: nonzero f_W obstructs local preparation

Obstruction

Let $\sigma = \Phi(\pi)$ with π a product state and Φ a local circuit with product ancillas and causal radius R. If $2(R + r) < s$, then $f_W(\sigma) = 0$.

Proof. Dilate Φ to a unitary U on system + ancillas, $\Pi = \pi \otimes \pi_{\mathrm{anc}}$. The pulled-back observables $O_i = U^\dagger(\widetilde{W_i} \otimes \mathbf{1})U$ live in disjoint radius-$(R + r)$ tubes, so

$$D_{12}(\sigma) = \mathrm{Tr}(\Pi O_1 O_2) = \mathrm{Tr}(\Pi O_1)\,\mathrm{Tr}(\Pi O_2) = M_1(\sigma)M_2(\sigma). \qquad \square$$

- approximate preparation: $|f_W(\rho) - f_W(\sigma)| \le 3\|\rho - \sigma\|_1$
- Hamiltonian or Lindblad evolution for time $o(L)$: Lieb–Robinson tails $\Rightarrow f_W \to 0$

$\limsup_L |f_W| > 0 \Rightarrow$ **not** in the trivial phase of product states ($T = \infty$ and $h \to \infty$ included).

Lieb & Robinson, CMP 28, 251 (1972); Bravyi, Hastings, Verstraete, PRL 97, 050401 (2006); H. Watanabe, arXiv:2607.00134

An invariant of channel phases: $\mathcal{F}$

A fixed decoder can fail along a trivial path (uniform on-site rotation $U(t)$: $f_W(1/2) = 0$ although the state stays in the same phase). So **optimize** over binary observables near the loops and physical recoveries with total radius $\leq \epsilon L$:

$$F_{\epsilon,L}(\rho) \coloneqq \sup_{(W_1,W_2,\mathcal{C})} |f_W(\rho;\mathcal{C})|, \qquad \mathcal{F}(\boldsymbol{\rho}) \coloneqq \lim_{\epsilon\downarrow 0} \limsup_{L\to\infty} F_{\epsilon,L}(\rho_L)$$

Phase invariant

$\mathcal{F}$ is nonincreasing under local channels with product ancillas and radius $o(L)$. Hence $\boldsymbol{\rho} \sim \boldsymbol{\sigma} \Rightarrow \mathcal{F}(\boldsymbol{\rho}) = \mathcal{F}(\boldsymbol{\sigma})$.

- $\mathcal{F} = 1$ for the canonical $T = 0$ field-free state; $\mathcal{F} = 0$ for every product state
- lower bound from the fixed decoder: $\mathcal{F} \geq \limsup_L |f_W|$, so $f_W \to 1$ implies $\mathcal{F} = 1$

$\mathcal{F}$ is constant on every two-way local-channel phase, and the measured $f_W \to 1$ forces $\mathcal{F} = 1$.

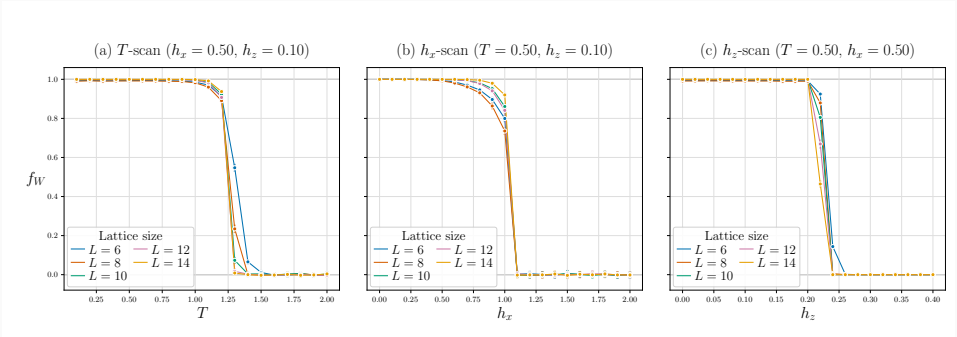
Why $\mathcal{F}$ is a phase invariant: proof

Let Φ_L be a local circuit with product ancillas and causal radius $R_L = o(L)$, and let $e_L := \|\sigma_L - \Phi_L(\rho_L)\|_1 \to 0$.

1. **Pull the candidate back.** For any admissible $(W_1, W_2, \mathcal{C})$ for σ_L: $f_W\big(\Phi_L(\rho_L); \mathcal{C}\big) = f_W\big(\rho_L; \mathcal{C} \circ \Phi_L\big)$, and $\mathcal{C} \circ \Phi_L$ is again a physical circuit with product ancillas whose radius grows by at most R_L.
2. **Conversion error.** $|f_W(\sigma_L; \mathcal{C}) - f_W(\Phi_L(\rho_L); \mathcal{C})| \le 3e_L$: covariances are trace-norm stable and $\mathcal{C}$ is contractive.
3. **Radius budget.** Fix a small $\epsilon > 0$; for large L, $R_L/L < \epsilon$, so
$$F_{\epsilon, L}(\sigma_L) \le F_{\epsilon + R_L/L,\, L}(\rho_L) + 3e_L \le F_{2\epsilon, L}(\rho_L) + 3e_L.$$
4. **Limits.** $\limsup_{L\to\infty}$ first (absorbs any sublinear enlargement), then $\epsilon \downarrow 0$: $\mathcal{F}(\boldsymbol{\sigma}) \le \mathcal{F}(\boldsymbol{\rho})$. A reverse conversion gives $\ge$. □

One-way local conversion cannot increase $\mathcal{F}$; two-way conversion fixes it. Product inputs give $\mathcal{F} = 0$ (disjoint backward cones), so $\mathcal{F} > 0$ excludes the trivial phase.

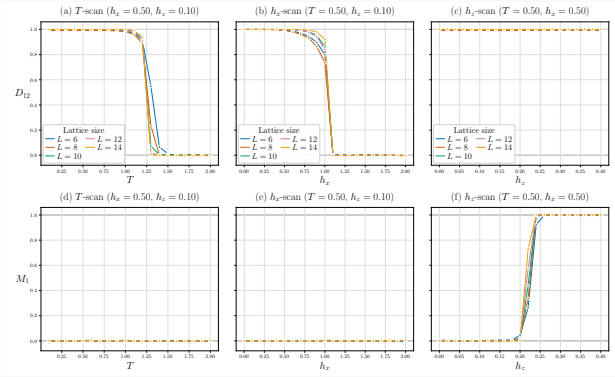
Quantum Monte Carlo: f_W along three cuts



- worldline QMC, equal-time Z snapshots, $L = 6, 8, 10, 12, 14$, fixed Sweep depth $r = \lfloor \sqrt{L} \rfloor$
- (a) T at $(h_x, h_z) = (0.5, 0.1)$; (b) h_x at $(T, h_z) = (0.5, 0.1)$; (c) h_z at $(T, h_x) = (0.5, 0.5)$

$f_W \simeq 1$ at low T and weak fields; $f_W \to 0$ beyond, sharpening with L.

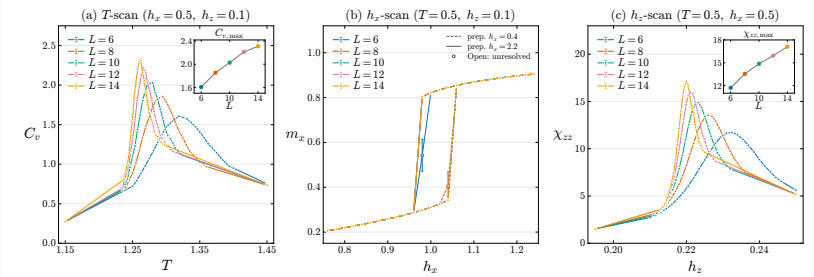
Two different ways to lose the bit



T and h_x scans: D_{12} drops while $M_i \simeq 0$
$\Rightarrow$ the two loop values stop agreeing

h_z scan: $D_{12} \simeq 1$ but $M_i \to 1 \Rightarrow$ polarization;
the subtraction is essential

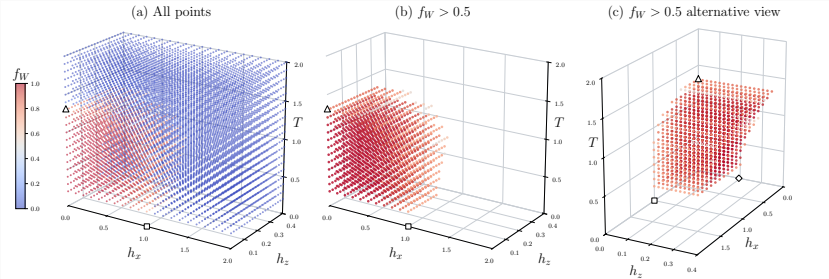
Thermodynamic signatures on the same cuts



- (a) the specific-heat peak and (c) the longitudinal-susceptibility peak grow with L where f_W drops
- (b) transverse polarization: preparation-dependent branches near the drop of f_W

The loss of f_W is accompanied by thermodynamic singularities.

The region with $\mathcal{F} = 1$



- $L = 8$ survey over (h_x, h_z, T): 8379 points; (b,c) the points with $f_W > 0.5$
- markers: known transitions at $(h_x, h_z, T) = (1, 0, 0),\ (0, 0.194, 0),\ (0, 0, 1.3133)$

An extended region at $T > 0,\ h_x, h_z > 0$ with $\mathcal{F} = 1$: a Gibbs-state phase **beyond microscopic symmetry breaking**.

Part II: summary, what $\mathcal{F}$ is not, and what is still missing

- $\mathcal{F} = 1$ and $\mathcal{F} = 0$ are **different phases** under two-way local channels; numerics give $f_W \to 1$, hence $\mathcal{F} = 1$, in a finite region of (T, h_x, h_z) with no microscopic higher-form symmetry.
- $\mathcal{F}$ is not quantized in general ($\mathcal{F} = 4p(1-p)$ for a mixture of two ferromagnets), $\mathcal{F} = 0$ does not prove triviality, and the bit is classical (three logical Z parities): no quantum memory is claimed.

Part II leaves open	$\Rightarrow$	**Part III**
two uniform fields h_x, h_z of one model; evidence by QMC	$\Rightarrow$	**arbitrary** weak local perturbations, any (d, m); theorems
thermodynamic connectivity: is f regular inside the $\mathcal{F} = 1$ region?	$\Rightarrow$	joint analyticity of f at low (and high) T
channel connectivity: only the one-way obstruction and $\mathcal{F} \geq \limsup \lvert f_W \rvert$; no two-way channel to a reference state	$\Rightarrow$	$\boldsymbol{\rho}_{\beta,g} \sim_{\mathrm{LC}} \boldsymbol{\rho}_{\mathrm{flat}}$ **both ways**: the whole domain is one phase, $\mathcal{F}_m = 1$

Closed flux loops + local decoding $\Rightarrow$ a thermal phase invariant beyond symmetry. Part III turns this picture into theorems.

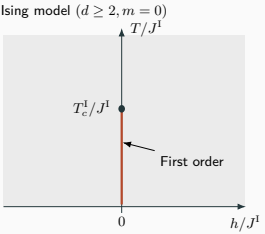
Part III: Classical $\mathbb{Z}_2$ models on m-cells — stability without symmetry

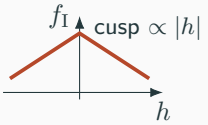
The Ising lesson

Ferromagnetic Ising model on a $d \geq 2$ lattice, $\sigma_i = \pm 1$:

$$H_{\mathrm{I}} = -J^{\mathrm{I}} \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i, \qquad J^{\mathrm{I}} > 0$$

$$f_{\mathrm{I}}(T,h) = f_{\mathrm{I}}(T,0) - m_{\mathrm{sp}}(T)\,|h| + o(|h|) \qquad (0 < T < T_c^{\mathrm{I}})$$



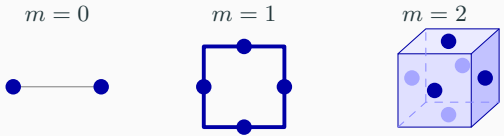


- a cusp at $h = 0$: a first-order line
- the ordered phase is protected by the $\mathbb{Z}_2$ symmetry
- any field $h \neq 0$ connects the ordered and disordered regimes smoothly

Can a nontrivial thermal phase survive **arbitrary** weak local perturbations with **no** symmetry protection?

Lee & Yang, Phys. Rev. 87, 410 (1952); Friedli & Velenik, *Statistical Mechanics of Lattice Systems* (2017); Ott, arXiv:2003.05879

A family of models: spins on m-cells



$$B_p := \prod_{\ell \in \partial p} Z_\ell, \qquad H_{d,m} := -J \sum_{p\,:\,(m+1)\text{-cells}} B_p$$

- periodic d-dimensional cubic lattice, full Hilbert space, no Gauss law
- $m = 0$: Ising model; $m = 1$: plaquette (= toric-code magnetic sector); $m = 2$: cube term
- defects ($B_p = -1$) are dual $(d - m - 1)$-cells; the Bianchi identity makes them **closed**: domain walls, loops, sheets, ...

Wegner, J. Math. Phys. 12, 2259 (1971); Kitaev (2003); Gaiotto *et al.*, JHEP 02, 172 (2015)

Two ingredients $\Rightarrow$ the stability range $1 \le m \le d-2$

(i) $m \ge 1$**: a nonlocal sector label.**
Flat sectors are labeled by Wilson products on winding m-surfaces; local gauge flips η_u on $(m-1)$-cells make all sectors **locally indistinguishable**. At $m = 0$ a single spin reads the sector $\Rightarrow$ the Ising cusp.

(ii) $k = d - m - 1 \ge 1$**: extended defects.**
A defect of r cells costs $2Jr$ while its shapes grow only exponentially $\Rightarrow$ low T wins; small components can be **filled locally**. At $m = d-1$ defects are points and $f = -T[d \log 2 + \log\cosh(J/T)]$ is analytic.



Nonlocal sector + extended closed defects = stability without symmetry.

Perturbations and the stability theorems

$H_g = H_0 + gV$, $V = \sum_X v_X$: $\|v_X\| \le 1$, support $\le b$ cells, diameter $\le R$, overlap $\le n_{\mathrm{inc}}$ — **arbitrary** local terms, classical or noncommuting quantum, no symmetry required (e.g. $V = -\sum_\ell Z_\ell$).

Theorem 1: low-temperature stability, $1 \le m \le d-2$

There are $B, a, C > 0$ such that the free-energy density $f(\beta, g)$ is jointly real-analytic in

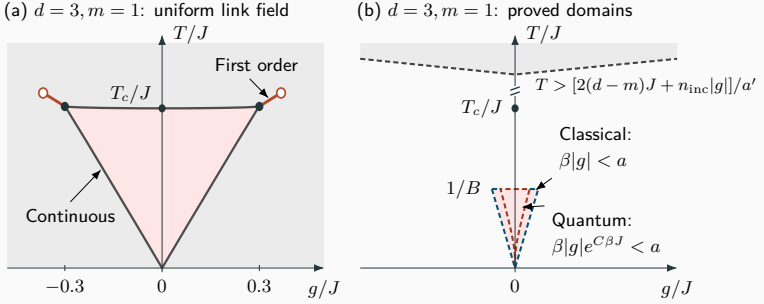
$$\beta J > B, \quad \beta|g| < a \ \text{(classical)}; \qquad \beta J > B, \quad \beta|g|\, e^{C\beta J} < a \ \text{(quantum)}.$$

Theorem 2: high-temperature analyticity, $0 \le m \le d-1$

f is jointly real-analytic whenever $\beta\big[2(d-m)J + n_{\mathrm{inc}}|g|\big] < a'$; this reaches $T = \infty$ at any fixed g.

No Ising-type cusp at $g = 0$: at low T the free energy is regular in **every** perturbation direction.

Sufficient domains for $(d, m) = (3, 1)$



Classical wedge, narrower quantum window, high-T domain reaching $T = \infty$.
Dashed lines are bounds, not phase boundaries; the white region is open.

(a): Somoza, Serna, Nahum, PRX 11, 041008 (2021); Bonati, Pelissetto, Vicari, PRB 105, 165138 (2022); Fradkin & Shenker (1979). (b): Sengoku, Cao, H. Watanabe, arXiv:2610.27938

Proof idea I: gauge averaging turns any perturbation into closed components

In a free box, H_0 is invariant under the local gauge flips $(\eta\sigma)_\ell = \sigma_\ell \prod_{u\in\partial\ell} \eta_u$, so exactly

$$Z^{\mathrm{o}} = \sum_{\sigma} e^{-\beta H_0^{\mathrm{o}}(\sigma)}\, \mathbb{E}_\eta\, e^{-\beta g V^{\mathrm{o}}(\eta\sigma)} \qquad \text{(average the Boltzmann weight, not } V\text{)}$$

- expand $e^{-\beta g c_X \sigma_X} = \cosh k_X\,(1 + \tanh k_X\, \sigma_X)$, $k_X := -\beta g c_X$: a factor $\tanh k_X$ per selected term
- the average kills every selection with uncanceled boundary $\Rightarrow$ only **closed** m-dimensional selections ξ survive, fillable inside a local box ($m \geq 1$)
- exact two-species polymer form, SM Eq. (S33): defect sets E and perturbation selections ξ are coupled only by a **crossing sign**

$$Z^{\mathrm{o}} = e^{-\beta g V_c^{\mathrm{o}}}\, 2^{z_m}\, e^{\beta J N_p} \prod_X \cosh k_X \sum_{\xi \text{ neutral}} \sum_{E \text{ closed}} \omega_{\mathrm{e}}(\xi)\, \omega_{\mathrm{f}}(E)\, \chi(\xi, E)$$

$$\omega_{\mathrm{e}}(\xi) = \prod_{X\in\xi} \tanh k_X, \qquad \omega_{\mathrm{f}}(E) = e^{-2\beta J|E|}, \qquad \chi(\xi, E) = \pm 1 \text{ (crossing sign)}$$

The cluster expansion of $\log Z^{\mathrm{o}}$ converges uniformly on complex neighborhoods $\Rightarrow$ analyticity of the exact f.

Arbitrary symmetry-breaking weights become closed, locally fillable components.

Proof idea II: noncommuting perturbations

Diagonal weight ratio for a product Z eigenvector $|\sigma\rangle$, with $v_X(u) = e^{u\beta H_0^o} v_X e^{-u\beta H_0^o}$:

$$\mathcal{W}_{\beta,g}(\sigma) := e^{\beta H_0^o(\sigma)} \langle\sigma| e^{-\beta H_g^o} |\sigma\rangle = \langle\sigma| \mathcal{T} \exp\Big[-\beta g \int_0^1 du \sum_X v_X(u) \Big] |\sigma\rangle$$

- H_0 commutes term by term $\Rightarrow$ each $v_X(u)$ stays within a bounded *enlarged support* (X plus the B_p meeting it), with $\|v_X(u)\| \le e^{C\beta J}$
- the connected expansion of $\log \mathcal{W}$ converges for small $\beta|g|e^{C\beta J}$: an effective **classical** interaction with exponentially summable tails
- the classical argument then applies with weighted bounds: no spectral gap, no truncation in g
- high temperature: a finite-range cluster expansion about $\beta = 0$, valid for noncommuting terms

Quantum perturbations reduce to a weak classical interaction; the price is the factor $e^{C\beta J}$ in the window.

Borgs, Kotecký, Ueltschi (1996); Kotecký & Ueltschi (1999); Fröhlich, Rey-Bellet, Ueltschi (2001); Harrow, Mehraban, Soleimanifar (2020)

Conjecture: thermal phase separation

Conjecture

For every $1 \leq m \leq d-2$ and every allowed V, any continuous (T, g) path with $T > 0$ from the low-temperature domain to the high-temperature domain crosses a thermodynamic nonanalyticity of f, even when T and g vary together.

- stronger than both theorems: they give two analytic islands, not a wall between them
- the number, location and order of transitions are not specified
- motivation: the two domains are distinguished at the level of **states** by local channels (next)

Analyticity on both sides + a channel invariant that differs
$\Rightarrow$ a transition is expected, but not yet proven.

Local-channel stability of the Gibbs states

Theorem 1: low-temperature local-channel stability, $1 \le m \le d-2$

There are $\tilde{B}, \tilde{a}, \tilde{C} > 0$ such that $\beta J > \tilde{B}$, $\beta|g| < \tilde{a}$ (V classical) or $\beta J > \tilde{B}$, $\beta|g|\, e^{\tilde{C}\beta J} < \tilde{a}$ (V Hermitian) implies $\boldsymbol{\rho}_{\beta,g} \sim_{\mathrm{LC}} \boldsymbol{\rho}_{\mathrm{flat}}$, with $\rho_{\mathrm{flat},L} = |\mathcal{A}_{\mathrm{flat}}|^{-1} \sum_{a\ \mathrm{flat}} |a\rangle\langle a|$ (all winding sectors). Product ancillas, polylogarithmic causal radius, whole-system trace error $O(L^{-s})$ ($s > 0$ fixed).

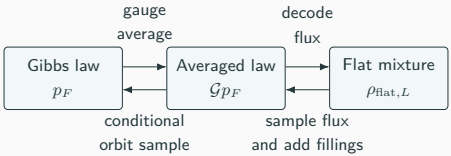
Theorem 2: high-temperature local preparation, $0 \le m \le d-1$ (same guarantees)

For some $\tilde{a}' > 0$: $\quad \beta\big[2(d-m)J + n_{\mathrm{inc}}|g|\big] < \tilde{a}' \ \Rightarrow\ \boldsymbol{\rho}_{\beta,g} \sim_{\mathrm{LC}} \boldsymbol{\rho}_{\infty} = 2^{-n} I.$

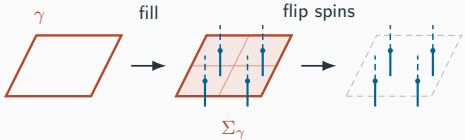
- $g = 0$ gives $\boldsymbol{\rho}_{\beta,0} \sim_{\mathrm{LC}} \boldsymbol{\rho}_{\mathrm{flat}}$; composing the conversions, $\boldsymbol{\rho}_{\beta,g} \sim_{\mathrm{LC}} \boldsymbol{\rho}_{\beta,0}$ at the same temperature
- no gap of H_g is assumed; weakly coupled physical ancillary systems can be included in the target state

The perturbed Gibbs state **is** the canonical flat ensemble up to local operations, in both directions.

Proof idea I: the construction



1. gauge average $\leftrightarrow$ conditional orbit sampling
2. averaged law $=$ weakly interacting **closed defect gas**
3. decode: fill small components locally (Peierls: $\Pr[|\gamma| \geq a_0 \log L] = O(L^{-s})$)
4. encode: sample dilute components from local coins
5. quantum V: dephase $\leftrightarrow$ restore the coherences by a local channel (next slide)



fill a flux loop γ by a dual surface Σ_γ;

flipping the piercing links removes exactly γ

Xun, Cao, H. Watanabe, arXiv:2610.27939; Bravyi & Hastings, CMP 307, 609 (2011); Bachmann, Michalakis, Nachtergaele, Sims, CMP 309, 835 (2012); Haah, Hastings, Kothari, Low (2021)

Proof idea II: the conversion channels in formulas

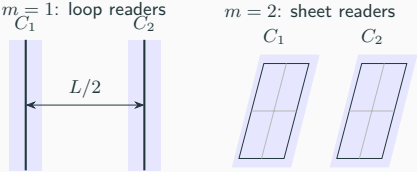
1. **Gauge average and its inverse**: $w_F(a) := 2^{-n_{m-1}} \sum_{\widehat{\eta}} e^{F(a \oplus \delta \widehat{\eta})}$ with $F := -\beta g V$; the inverse samples $Q_a(\widehat{\eta}) = e^{F(a \oplus \delta \widehat{\eta})} / [2^{n_{m-1}} w_F(a)]$ by local heat-bath updates
2. **Closed defect gas**: $\log w_F(a) = c_L - U(E(a)) + r_L(a)$, local strength of U at most $C\beta|g|$ with exponential tails, $\|r_L\|_\infty \leq Cn\beta|g|e^{-cL}$
3. **Decode**: $\mathbb{P}(\exists \gamma : |\gamma| \geq r) \leq 2Q_L(r)$, $Q_L(r) = n_{m+1} e^{-\tau r}/(1 - e^{-\tau})$, $\tau = 2\beta J - 2 \log D_{d,m}$; cutoff $r = \lceil a_0 \log L \rceil \Rightarrow$ error $O(L^{-s})$
4. **Encode**: birth–death sampler with insertion rate $b_\gamma(E) = e^{-2\beta J|\gamma| - U_r(E \cup \gamma) + U_r(E)}$, converging in $O(\log L)$ time
5. **Quantum** V: dephased weights $\mathcal{W}_{\beta,g}(a) = e^{\beta H_0(a)} \langle a | e^{-\beta H_g} | a \rangle$ (small $\lambda = \beta|g| e^{4b(d-m)\beta J}$); coherences restored by spectral flow of the gapped parent $\mathcal{K}_a(t) = \sum_\ell Q_\ell^\dagger Q_\ell$, $Q_\ell = M_t P_\ell(a) M_t^{-1}$, which maps $|a\rangle \mapsto |\psi_{1,a}\rangle \propto e^{-\beta H_g/2} |a\rangle$

Every step is a local circuit with product work registers: errors and radii add.

Xun, Cao, H. Watanabe, arXiv:2610.27939; Fernández, Ferrari, Garcia (2002); White, PRL 102, 190601 (2009); Feiguin & Klich (2013); Bravyi & Hastings (2011); Bachmann *et al.* (2012); Haah *et al.* (2021)

The invariant $\mathcal{F}_m$ and the phase distinction

Higher-cell version of Part II's $\mathcal{F}$ ($\mathcal{F}_1 = \mathcal{F}$ for the 3D toric code; recoveries absorbed into the readers): binary readers $A_i^2 = I$ on neighborhoods $R_{i,L}(\zeta)$ of parallel winding m-tori C_1, C_2 separated by κL:



$$\mathcal{F}_m(\boldsymbol{\rho}) \coloneqq \lim_{\zeta \downarrow 0} \limsup_{L \to \infty} \sup_{A_i} \left| \langle A_1 A_2 \rangle_\rho - \langle A_1 \rangle_\rho \langle A_2 \rangle_\rho \right|$$

- monotone under one-way, invariant under two-way conversion
- flat ensemble: $W_{C_1} = W_{C_2}$, $\langle W_{C_i} \rangle = 0 \Rightarrow \mathcal{F}_m = 1$; product state: $\mathcal{F}_m = 0$
- mutual information: $\mathcal{F}_m \leq \sqrt{2I(R_1{:}R_2)}$ (Pinsker); $\mathcal{F}_m = 1$ forces $I(R_1{:}R_2) \geq \ln 2$

$$\mathcal{F}_m(\boldsymbol{\rho}_{\beta,g}) = \begin{cases} 1, & \text{low-temperature domains} \\ 0, & \text{high-temperature domain} \end{cases} \qquad \begin{array}{l} \text{for every } 1 \leq m \leq d-2 \\ \text{and every allowed } V \end{array}$$

Part III: summary

- A two-state cell family containing the Ising model ($m = 0$) and the toric-code magnetic sector ($m = 1$)
- Stability range $1 \leq m \leq d-2$: **nonlocal sector label + extended closed defects**
- Proven: joint analyticity of f at low T under arbitrary weak local (classical or quantum) perturbations; high-T analyticity; two-way local-channel equivalence in both regimes; $\mathcal{F}_m$ separates them
- Open: sharp boundaries, the number and order of transitions, $T \to 0$ at fixed g, noncommuting logical information

Stable nontrivial thermal phases exist with **no** symmetry protection, in every dimension $d \geq 3$.

Conclusion

Summary

	spin ice / dense ice	3D toric code in fields	$\mathbb{Z}_2$ cell models, $1 \leq m \leq d-2$
defect constraint	energetic ice rule	Bianchi identity	Bianchi identity
can loops break?	**yes** (monopoles)	**no**	**no**
sector label	global flux (lost at $T > 0$)	winding Wilson loops	Wilson m-surfaces
fate at $T > 0$	crossover	$\mathcal{F} = 1$ region (QMC)	proven stable windows
classification	symmetry suffices	beyond symmetry	beyond symmetry

Beyond symmetry breaking at $T > 0$: a **nonlocal sector label** that bounded regions cannot read, carried by **closed extended defects** that local operations can clean up.

Outlook

- **Thermal phase separation**: prove that a thermodynamic singularity separates the two analytic domains
- **Sharp boundaries**: where does $\mathcal{F}$ jump?
  relation between channel phases and free-energy analyticity
- **Experiment**: realizing the 3D toric code and its thermal phase
- **Beyond**: searching for other finite-temperature topological phases

Thank you!

I. S. Watanabe, Y. Motome, H. Watanabe, PRB 114, 094108 (2026); PRL in print (arXiv:2603.03852); PRX 16, 041016 (2026); JPSJ 95, 085002 (2026)

II. H. Watanabe, arXiv:2607.00134

III. S. Sengoku, W. Cao, H. Watanabe, arXiv:2610.27938; G. Xun, W. Cao, H. Watanabe, arXiv:2610.27939